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HOW MANY INTERVALS COVER A POINT IN DVORETZKY COVERING?

BY

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ABSTRACT

Consider the Dvoretzky random covering with length sequence $\{\alpha/n\}_{n\geq 1}$ ($\alpha > 0$). We are interested in the set F_β of points on the circle which are covered by a number $\beta \log n$ of the first n randomly placed intervals. It is proved among others that for a certain interval of $\beta > 0$, the Hausdorff dimension of F_β is equal to $1 - [\beta\log(\beta/\alpha) - (\beta - \alpha)]$. This implies that points on the circle are differently covered.

1. Introduction

We consider the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ which is identified with the interval $[0, 1)$, a decreasing sequence $\{\ell_n\}_{n\geq 1}$ ($0 < \ell_n < 1$) such that $\sum_n \ell_n = \infty$ and an i.i.d. random sequence $\{\omega_n\}_{n\geq 1}$ of the uniform distribution (Lebesgue distribution). We denote by I_n (or more precisely $I_n(\omega)$) the open interval of length ℓ_n, with left end point ω_n. The Dvoretzky covering problem is to give necessary conditions and sufficient conditions on the length sequence (ℓ_n) in order to cover the whole circle $\mathbb{T}$ almost surely (a.s. for short), or equivalently to have

$$P\left(\mathbb{T} = \bigcup_{n=1}^{\infty} I_n\right) = 1$$

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where P is the probability measure of the underlying probability space $(\Omega, \mathcal{A}, P)$.

The problem was raised in 1956 by A. Dvoretzky [D]. It attracted the attention of P. Lévy, J. P. Kahane, P. Erdös and P. Billard who made significant contributions (see [K1]). In 1972, L. Shepp [S1] gave a complete solution to the problem by finding a necessary and sufficient condition

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \exp(\ell_1 + \cdots + \ell_n) = \infty.$$

To get more information on further developments of the subject, we refer to J. P. Kahane's book [K1] and his nice survey papers [K4, K5].

From the Shepp condition, we see that when the whole circle is covered, every point on the circle is covered by an infinite number of intervals. L. Carleson then asked the question: How many intervals cover one point? (personal communication to J. P. Kahane). This question was partially answered in [FK1] in the case $\ell_n = \alpha/n$ with $\alpha > 1$: Let

$$N_n(t) = \mathrm{Card}\{1 \leq j \leq n : I_n \ni t\}.$$

There are two constants A_α, B_α depending on α with $0 < A_\alpha \leq \alpha \leq B_\alpha < \infty$ such that almost surely for *every* $t \in \mathbb{T}$, we have

$$A_\alpha \leq \liminf_{n\to\infty} \frac{N_n(t)}{\log n} \leq \limsup_{n\to\infty} \frac{N_n(t)}{\log n} \leq B_\alpha.$$

A nearly trivial result (see also [FK1] for explanation) is that almost surely for *almost every* $t \in \mathbb{T}$ (with respect to Lebesgue measure), we have

$$\lim_{n\to\infty} \frac{N_n(t)}{\log n} = \alpha.$$

Having this information, we would like to raise some natural questions. We take a positive number β which may be different from α and we consider the (random) set

$$F_\beta = \left\{ t \in \mathbb{T} : \lim_{n\to\infty} \frac{N_n(t)}{\log n} = \beta \right\}.$$

Is the set F_β non-empty for some $\beta \neq \alpha$? How big is the set F_β? How rich is the set of β's such that F_β is non-empty? We say that points in F_β are β-regularly covered. We say that $t \in \mathbb{T}$ is irregularly covered if

$$\liminf_{n\to\infty} \frac{N_n(t)}{\log n} < \limsup_{n\to\infty} \frac{N_n(t)}{\log n}.$$

Are there irregularly covered points? How many are there?

In this paper we partially answer these questions by showing the following two theorems. We will use $\dim E$ to denote the Hausdorff dimension of a set E.

THEOREM 1: *Let $\ell_n = \alpha/n$ with $\alpha > 0$.*

(a) *If $\beta \in (\alpha - \sqrt{\alpha}, \alpha + \sqrt{\alpha}) \bigcap (0, \infty)$, we have almost surely*

$$\dim F_\beta \geq 1 - \frac{|\beta - \alpha|^2}{\alpha} > 0.$$

(b) *If $\beta \log(\beta/\alpha) - (\beta - \alpha) \leq 1$, we have almost surely*

$$\dim F_\beta \leq 1 - [\beta \log(\beta/\alpha) - (\beta - \alpha)].$$

(c) *If $\max(\alpha - 1, 0) < \beta < \alpha$, we have almost surely*

$$\dim F_\beta = 1 - [\beta \log(\beta/\alpha) - (\beta - \alpha)].$$

We remark that the statement in part (b) of Theorem 1 holds for the packing dimension of F_β, which is what we actually prove. As a consequence, in part (c) of Theorem 1, the Hausdorff dimension can be replaced by the packing dimension.

THEOREM 2: *Let $\ell_n = \alpha/n$ with $\alpha > 0$. Almost surely, the set of irregularly covered points is of Hausdorff dimension* 1.

One of the main ideas is to consider the following formally defined random measure (weak* limit)

$$Q^a(dt) = \prod_{n=1}^{\infty} \frac{a^{1_{(0,\ell_n)}(t-\omega_n)}}{1 + (a-1)\ell_n} dt \quad (\text{with } a = \beta/\alpha)$$

and to show that the measure Q^a is supported by F_β. Such a measure is called a multiplicative chaos. We are then led to estimate the dimension of the measure Q^a. It is more practical for us to work with a variant of Q^a, called a Poisson multiplicative chaos, that we denote by P^a. We say that P^a is a variant because P^a is a.s. equivalent to Q^a (Theorem 4). Therefore $\dim Q^a = \dim P^a$ a.s. On the other hand, we calculate directly $\dim P^a$ (Theorem 3).

In §2, we introduce the notions of multiplicative chaos and dimension of measure. Theorem 1 (a)–(b) and Theorem 2 are respectively proved in §3 and §4. The more difficult part is Theorem 1 (c) whose proof occupies the rest of the paper.

We point out that a study on dyadic random covering is carried out in [FK2]. The results obtained there are more complete.

2. Tools

We need two tools. The first one is the infinite product of processes studied in a general setting by J. P. Kahane [K2]. We briefly present it here for the case adapted to our purpose. The key part for us is the Peyrière probability measure. Let (f_n) be a sequence of non-negative Borel functions on $\mathbb{T}$ whose integrals with respect to Lebesgue measure are all equal to 1. Consider the (random) measures

$$Q_n(t)dt := \prod_{j=1}^{n} f_j(t-\omega_j)dt.$$

It was proved that a.s. the sequence of measures $Q_n(t)dt$ converges weakly to a (random) measure, called a **multiplicative chaos**, which we denote by Q. The partial product sequence $Q_n(t)$ is called an **indexed martingale**, because it is a martingale for each t. If

$$\int\int\prod_{j=1}^{n}\mathbb{E}f_j(t-\omega_j)f_j(s-\omega_j)dtds = \int\int\prod_{j=1}^{n} f_j * \check{f}_j(t-s)dtds$$

is bounded as $n\to\infty$ (where $\check{f}(t)=f(-t)$), the martingale $\int_0^1 Q_n(t)dt$ converges in L^2 and the measure Q does not vanish and a probability measure $\mathcal{Q}$ on $\Omega\times\mathbb{T}$, called the **Peyrière measure**, can be defined by the relation

$$\int_{\Omega\times\mathbb{T}}\varphi(\omega,t)d\mathcal{Q}(\omega,t) = \mathbb{E}\int\varphi(\omega,t)dQ(t)$$

(for all bounded measurable functions φ). A very useful fact is that $X_n = t-\omega_n$ ($n\geq 1$), considered as random variables on $\Omega\times\mathbb{T}$, are $\mathcal{Q}$-independent. Furthermore, we have the formula

$$\mathbb{E}_{\mathcal{Q}}h(t-\omega_n) = \int_{\mathbb{T}} h(u)f_n(u)du$$

(for any positive or bounded Borel function h).

The second tool is the following principle for estimating the dimension of a set [F2]. The **energy integral** of order τ ($0<\tau<1$) of a measure μ on $\mathbb{T}$ or $\mathbb{R}$ is defined by

$$I_\tau^\mu = \int\int\frac{d\mu(t)\ d\mu(s)}{|t-s|^\tau}.$$

The **(lower) dimension** of a measure μ, denoted by $\dim\mu$, is the supremum of β's such that $\mu(E)=0$ for any E with $\dim E<\beta$ [F2]. So, for a given set F, we have $\dim F\geq\dim\mu$ if $\mu(F)>0$. To estimate the dimension of a measure, we shall use the fact that $\dim\mu \geq \tau$ if $I_\tau^\mu < \infty$ [F2]. In general, we have the formula [F2]

$$\dim\mu = \sup\left\{\gamma \geq 0 : \liminf_{r\to 0} \frac{\log\mu(B_r(x))}{\log r} \geq \gamma \quad \mu\text{-a.e.}\right\}$$

where $B_r(x)$ denotes the interval centered at x of length $2r$. There is also a notion of upper dimension, but it is not needed here.

Notation: For a sequence of real numbers a_n and a sequence of positive numbers b_n, $a_n = O(b_n)$ means $|a_n| \leq Cb_n$ for some constant $C > 0$; $a_n \approx b_n$ means $a_n = O(b_n)$ and $b_n = O(a_n)$; $a_n \sim b_n$ means $a_n/b_n \to 1$.

3. Proofs of Theorem 1 (a) and (b)

Proof of Theorem 1 (a): For $a > 0$, consider the random measure Q^a defined by the indexed martingale

$$Q_n^a(t) = \prod_{j=1}^n \frac{a^{\chi_j(t-\omega_j)}}{1+\alpha(a-1)/j}$$

where χ_j is the characteristic function of the interval $(0, \alpha/j)$. Take a such that $\alpha(1-a)^2 < 1$. The corresponding indexed martingale is a L^2 martingale and then gives rise to a non-vanishing random measure Q^a [FK1]. That Q^a does not vanish is a tail event because the j-th factor in $Q_n^a(t)$ is bounded from below by c_j and from above by C_j where the two constants $c_j > 0$ and $C_j > 0$ can be chosen to be

$$c_j = \frac{\min(a,1)}{1+(a-1)\ell_j}, \quad C_j = \frac{\max(a,1)}{1+(a-1)\ell_j}.$$

So, a.s. we have $Q^a \neq 0$. Let $\mathcal{Q}^a$ be the associated Peyrière measure. We claim that, for any $\eta > 1/2$, $\mathcal{Q}^a$-almost surely (then almost surely Q^a-almost everywhere)

$$\sum_{j=1}^n \chi_j(t-\omega_j) = a\alpha\log n + O(\log^\eta n) \quad (n\to\infty). \tag{3.1}$$

We postpone the proof of (3.1). It follows from (3.1) that almost surely

$$Q^a(F_{a\alpha}^c) = 0.$$

That means $F_{a\alpha}$ is of full Q^a-measure. Thus we have almost surely

$$\dim F_{a\alpha} \geq \dim Q^a.$$

On the other hand, consider the energy integral of Q^a of order τ:

$$I_\tau^{Q^a} = \int\int \frac{dQ^a(t)\, dQ^a(s)}{|t-s|^\tau}.$$

By the calculation in [FK1], it may be proved that $\mathbb{E} I_\tau^{Q^a} < \infty$ if

$$\int\int \frac{dt\, ds}{|t-s|^{\alpha(a-1)^2+\tau}} < \infty.$$

The finiteness of the last integral is guaranteed by $\alpha(a-1)^2 + \tau < 1$, in other words, $\tau < 1 - \alpha(a-1)^2$. Then, under this condition on τ, $I_\tau^{Q^a}$ is almost surely finite. This implies that

$$\dim Q^a \geq 1 - \alpha(a-1)^2.$$

To obtain the claim in Theorem 1 (a), it suffices to take $a = \beta/\alpha$ for a given β.

We now prove (3.1). Consider

$$S_n = \sum_{j=1}^n Y_j, \quad Y_j = X_j - \mathbb{E}_{Q^a} X_j$$

where $X_j = \chi_j(t-\omega_j)$ $(j \geq 1)$ are Q^a-independent variables (see §2). We first estimate the variance of S_n. Notice that

$$\begin{aligned}\mathbb{E}_{Q^a} X_j =& \mathbb{E}\chi_j(t-\omega_j)\frac{a^{\lambda_j(t-\omega_j)}}{1+\alpha(a-1)/j} \\ =& \frac{a\alpha/j}{1+\alpha(a-1)/j} = \frac{a\alpha}{j} + O\Big(\frac{1}{j^2}\Big).\end{aligned}$$

Since X_j takes just two values 0 and 1, we have $\mathbb{E}_{Q^a} X_j^2 = \mathbb{E}_{Q^a} X_j$. Hence

$$\mathbb{E}_{Q^a} Y_j^2 = \mathbb{E}_{Q^a} X_j (1 - \mathbb{E}_{Q^a} X_j) = \frac{a\alpha}{j} + O\Big(\frac{1}{j^2}\Big).$$

Then

$$\mathbb{E}_{Q^a} S_n^2 = O(\log n). \tag{3.2}$$

Next, by the Kolmogorov inequality and (3.2),

$$Q^a\big(\max_{1\leq n\leq N} |S_n| \geq \lambda\big) \leq \frac{\mathbb{E}_{Q^a} S_N^2}{\lambda^2} = O\Big(\frac{\log N}{\lambda^2}\Big)$$

holds for any $N \geq 1$ and any $\lambda > 0$. Apply this inequality to $N = [\exp k^{\Delta}]$ and $\lambda = k^{\eta\Delta}$ with $\Delta = 2/(2\eta - 1)$, where $[x]$ denotes the integral part of a real number x. We get

$$\mathcal{Q}^a\big(\max_{1\leq n\leq \exp k^{\Delta}} |S_n| \geq k^{\eta\Delta}\big) = O\Big(\frac{k^{\Delta}}{k^{2\eta\Delta}}\Big) = O\Big(\frac{1}{k^2}\Big).$$

According to the Borel–Cantelli lemma, $\mathcal{Q}^a$-almost surely (then almost surely Q^a-almost everywhere)

$$\max_{1\leq n\leq \exp k^{\Delta}} |S_n| = O(k^{\eta\Delta}). \tag{3.3}$$

Suppose $\exp(k-1)^{\Delta} < n \leq \exp k^{\Delta}$. We have $(k-1)^{\Delta} < \log n$. So, by (3.3),

$$|S_n| = O(k^{\eta\Delta}) = O(\log^{\eta} n).$$

Thus (3.1) is proved for $\sum_{j=1}^{n} \mathbb{E}_{\mathcal{Q}^a} X_j = a\alpha \log n + O(1)$. ∎

Proof of Theorem 1 (b): Fix a dyadic interval $I = [\frac{i}{2^j}, \frac{i+1}{2^j}]$ $(0 \leq i < 2^j)$ of length 2^{-j}. Let

$$S_j(t) = \sum_{k=1}^{2^j} \chi_k(t - \omega_k).$$

Then let

$$X_I = \min_{t\in I} S_j(t), \quad Y_I = \max_{t\in I} S_j(t).$$

We will distinguish two cases: $\beta > \alpha$ and $\beta < \alpha$. To deal with the first (resp. second) case, we will work with Y_I (resp. X_I). Suppose $\beta > \alpha$. For any small $\delta > 0$, let

$$\mathcal{C}_j = \{I : |I| = 2^{-j}, Y_I \geq (\beta - \delta) j \log 2\},$$
$$M_j = \operatorname{card} \mathcal{C}_j = \sum_I 1_{\{Y_I \geq (\beta-\delta) j \log 2\}},$$
$$G_j = \bigcup_{I\in\mathcal{C}_j} I.$$

It is clear that

$$F_\beta \subset \bigcup_{\ell=1}^{\infty} \bigcap_{j=\ell}^{\infty} G_j.$$

It follows that

$$\dim_P F_\beta \leq \sup_{\ell\geq 1} \dim_P \bigcap_{j=\ell}^{\infty} G_j \leq \sup_{\ell\geq 1} \overline{\dim}_B \bigcap_{j=\ell}^{\infty} G_j$$

where $\dim_P$ denotes the Packing dimension and $\overline{\dim}_B$ denotes the upper box dimension (see [Mat] for their definitions). However, $\mathcal{C}_j$ is a 2^{-j}-cover of $\bigcap_{j=\ell}^{\infty} G_j$ when $j \geq \ell$. So we have

$$\overline{\dim}_B \bigcap_{j=\ell}^{\infty} G_j \leq \lim_{j\to\infty} \frac{\log M_j}{\log 2^j}.$$

Then we are led to estimate the random variable M_j. Take an α^* such that $0 < \alpha < \alpha^* < \beta - \delta$. We consider a covering with enlarged random intervals. Denote

$$S_j^*(t) = \sum_{k=1}^{2^j} \chi_k^*(t - \omega_k)$$

where χ_k^* is the characteristic function of the interval with the same center as $(0, \alpha/k)$ but of length α^*/k. Let I^- (resp. I^+) be the left (resp. right) dyadic interval of length 2^{-j} next to I. Consider the random set

$$E_I := \{t \in I^- \bigcup I \bigcup I^+ \colon S_j^*(t) \geq Y_I\}.$$

Suppose $Y_I = S_j(t_0) = N$. That means there is a (random) point t_0 in the interval I which is covered exactly N times. If the interval I_k covers t_0, the corresponding enlarged interval must cover the interval centered at t_0 of length $(\alpha^* - \alpha)/k$. It follows that every point in the interval $(t_0 - \frac{\alpha^*-\alpha}{2\cdot 2^j}, t_0 + \frac{\alpha^*-\alpha}{2\cdot 2^j})$ is covered by at least N enlarged intervals. So we have

$$|E_I| \geq \frac{\alpha^* - \alpha}{2^j}.$$

By the definition of E_I, we have for $\lambda > 0$

$$e^{\lambda Y_I}|E_I| \leq \int_{E_I} e^{\lambda S_j^*(t)} dt \leq \int_{I^- \bigcup I \bigcup I^+} e^{\lambda S_j^*(t)} dt.$$

Then, by taking expectation, we get

$$\begin{aligned} \mathbb{E} e^{\lambda Y_I} \leq & |I^- \bigcup I \bigcup I^+| \mathbb{E} \frac{e^{\lambda S_j^*(0)}}{|E_I|} \\ \leq & \frac{3}{\alpha^* - \alpha} \prod_{k=1}^{2^j} \left(1 + (e^\lambda - 1)\frac{\alpha^*}{k}\right). \end{aligned}$$

Consequently, by using the Markov inequality, we get

$$\begin{aligned} P(Y_I \geq (\beta - \delta) j \log 2) \leq & \frac{\mathbb{E} e^{\lambda Y_I}}{2^{j(\beta-\delta)\lambda}} \\ \leq & \frac{3\exp((e^\lambda - 1)\alpha^*)}{\alpha^* - \alpha} \cdot 2^{-jh(\lambda)} \end{aligned}$$

where

$$h(\lambda) = (\beta - \delta)\lambda - \alpha^*(e^\lambda - 1).$$

We remark that the function $h(\lambda)$ is maximized at $\lambda = \log \frac{\beta-\delta}{\alpha^*} > 0$ and its maximal value is equal to

$$h_{\max} = (\beta - \delta) \log \frac{\beta - \delta}{\alpha^*} - (\beta - \delta - \alpha^*).$$

Thus we get

$$\mathbb{E}M_j = \sum_I P(Y_I \geq (\beta - \delta)j \log 2) \leq \frac{3e^{\beta-\alpha^*-\delta}}{\alpha^* - \alpha} \cdot 2^{j(1-h_{\max})}.$$

Consequently, for any $\eta > 0$,

$$\mathbb{E}\sum_j 2^{-j(1-h_{\max}+\eta)} M_j < \infty.$$

Then almost surely for any $\ell \geq 1$, we have

$$\overline{\dim}_B \bigcap_{j=\ell}^{\infty} G_j \leq \limsup_{j\to\infty} \frac{\log M_j}{\log 2^j} \leq 1 - h_{\max} + \eta.$$

Let successively $\alpha^* \to \alpha$, $\delta \to 0$, $\eta \to 0$. We get

$$\dim_P F_\beta \leq 1 - [\beta \log(\beta/\alpha) - (\beta - \alpha)].$$

Suppose now $\beta < \alpha$. The proof is almost the same. Let us just point out the minor differences we should make. For any small $\delta > 0$, let

$$\mathcal{C}_j = \{I : |I| = 2^{-j}, X_I \leq (\beta + \delta)j \log 2\},$$
$$M_j = \operatorname{Card} \mathcal{C}_j = \sum_I 1_{\{X_I \leq (\beta+\delta)j \log 2\}}$$

("$\geq$" is replaced by "$\leq$" and "$-\delta$" is replaced by "$+\delta$").

In order to estimate the random variable M_j, we take an α^{**} such that $0 < \beta + \delta < \alpha^{**} < \alpha$ and consider a covering with shortened random intervals. Let

$$S_j^{**}(t) = \sum_{k=1}^{2^j} \chi_k^{**}(t - \omega_k)$$

where χ_k^{**} is the characteristic function of the interval with the same center as $(0, \alpha/k)$ but of length α^{**}/k. It may be proved that for any $\lambda > 0$ we have

$$\mathbb{E}e^{-\lambda X_I} \leq \frac{3}{\alpha - \alpha^{**}} \prod_{k=1}^{2^j} \Big(1 + (e^{-\lambda} - 1)\frac{\alpha^{**}}{k}\Big).$$

Consequently, by using the Markov inequality, we get

$$\begin{aligned} P(X_I \le (\beta+\delta) j \log 2) &\le \frac{\mathbb{E}e^{-\lambda Y_I}}{2^{-j(\beta+\delta)\lambda}} \\ &\le \frac{3\exp((e^{-\lambda}-1)\alpha^{**})}{\alpha-\alpha^{**}} \cdot 2^{-jh(\lambda)} \end{aligned}$$

where

$$h(\lambda) = -(\beta+\delta)\lambda - \alpha^{**}(e^{-\lambda}-1).$$

The function $h(\lambda)$ is maximized at $\lambda = \log \frac{\alpha^{**}}{\beta+\delta} > 0$. ∎

4. Proof of Theorem 2

Take a sequence $a = (a_n)_{n\ge1}$ of positive numbers such that $\delta = \sup_{j\ge1} |a_j - 1| < \infty$. Note that we have

$$\sum_n \frac{a_n^2}{n^2} < \infty.$$

Consider the random measure Q^a, which is a variant of the measure used above, defined by the indexed martingale

$$Q_n^a(t) = \prod_{j=1}^n \frac{a_j^{\chi_j(t-\omega_j)}}{1+\alpha(a_j-1)/j}.$$

If $\alpha\delta^2 < 1$, the corresponding indexed martingale also gives rise to an almost surely non-vanish random measure Q^a. Denote again by $\mathcal{Q}^a$ the associated Peyrière measure. In the same way as we prove (3.1), for any $\eta > 1/2$ we can prove that almost surely Q^a-almost everywhere

$$\sum_{j=1}^n \chi_j(t-\omega_j) = \alpha \sum_{j=1}^n \frac{a_j}{j} + O(\log^\eta n) \quad (n\to\infty).$$

Take now $0<\delta<1$, a number sufficiently small such that $\alpha\delta^2 < 1$, and a rapidly increasing sequence of integers (γ_j) such that

$$\gamma_1 + \cdots + \gamma_k = o(\gamma_{k+1}) \quad (k\to\infty).$$

Next let $(n_k)_{k\ge0}$ be the sequence of integers defined by

$$n_0 = 1, \quad 2^{-\gamma_{k+1}} = \frac{n_0+n_1+\cdots+n_k}{n_0+n_1+\cdots+n_{k+1}}.$$

Then define a sequence (a_n) as follows:

$$a_n = 1 - \delta \quad \text{if } n_0 + \cdots + n_{2k} < n \leq n_0 + \cdots + n_{2k+1},$$
$$a_n = 1 + \delta \quad \text{if } n_0 + \cdots + n_{2k-1} < n \leq n_0 + \cdots + n_{2k}.$$

For this choice of sequence (a_n), we get

$$\sum_{j=1}^{n_o+\cdots+n_{2k}} \frac{a_j}{j} = (1+\delta)\log\frac{n_0+\cdots+n_{2k}}{n_0+\cdots+n_{2k-1}} + O(\gamma_1+\cdots+\gamma_{2k-1}).$$

It follows that

$$\lim_{k\to\infty}\frac{1}{\log(n_0+\cdots+n_{2k})}\sum_{j=1}^{n_0+\cdots+n_{2k}}\frac{a_j}{j} = 1+\delta.$$

Similarly, we have

$$\lim_{k\to\infty}\frac{1}{\log(n_0+\cdots+n_{2k-1})}\sum_{j=1}^{n_0+\cdots+n_{2k-1}}\frac{a_j}{j} = 1-\delta.$$

So we have that almost surely Q^a-almost everywhere

$$\liminf_{n\to\infty}\frac{N_n(t)}{\log n} \leq (1-\delta)\alpha < (1+\delta)\alpha \leq \limsup_{n\to\infty}\frac{N_n(t)}{\log n}.$$

The energy integral of Q^a of order τ can be computed to be finite if $\alpha\delta^2 + \tau < 1$. So the dimension of the irregularly covered points is bounded from below:

$$\dim Q^a \geq 1 - \alpha\delta^2 \to 1 \quad (\delta \to 0).$$

5. Proof of Theorem 1 (c), Poisson multiplicative chaos

The rest of the paper is devoted to the proof of Theorem 1 (c). In this section we just present the strategy of the proof. Details will be given in the next sections.

As was seen in the proof of Theorem 1 (a), we have only to show that a.s.

$$\dim Q^a \geq 1 - [\beta\log(\beta/\alpha) - (\beta-\alpha)]$$

where $a = \beta/\alpha$ (see the formula at the end of §2). Instead of Q^a, we will work with a Poisson multiplicative chaos P^a, because its dimension is easier to calculate (see the following Theorem 3). We will also prove the equivalence of Q^a and P^a (see Theorem 4), which implies that Q^a and P^a have the same dimension [F2]. In this way, Theorem 1 (c) will be proved.

Now let us give more details of the strategy.

Define first the Poisson multiplicative chaos P^a. Let $\lambda = dx$ be the Lebesgue measure on $\mathbb{R}$ and let μ be a measure on $\mathbb{R}^+ = (0, +\infty)$ which is assumed finite on compact sets and concentrated on the interval $(0, 1)$. Consider the product measure $\nu = \lambda \otimes \mu$ on $\mathbb{R} \times \mathbb{R}^+$ and then the Poisson point process (X_n, Y_n) with intensity ν. A corresponding Poisson covering problem was considered in [Man,S2] and its relation with the Dvoretzky covering was revealed by Kahane [K3]. For $t \in \mathbb{R}$, denote

$$D_t = \{(x, y) \in \mathbb{R} \times \mathbb{R}^+ : y > 0, t - y < x < t\}.$$

For a fixed positive number $0 < a < \infty$, construct an indexed martingale

$$P_\epsilon^a(t) = a^{N_\epsilon(t)} \exp[(1-a)\nu_\epsilon(D_t)] \quad (t \in \mathbb{R},\ \epsilon > 0)$$

where ν_ϵ is the truncation of ν defined by $\lambda \otimes \mu_\epsilon$ with $\mu_\epsilon = \mu 1_{[\epsilon,\infty)}$, the restriction of μ on $[\epsilon, \infty)$ and $N_\epsilon(t)$ is the number of points in the domain D_t of the Poisson process with intensity ν_ϵ. By discretizing ϵ by a decreasing sequence $\{\epsilon_n\}$, we may write $P_{\epsilon_n}^a$ as a product of independent variables $P_{\epsilon_j}^a / P_{\epsilon_{j-1}}^a$ ($\epsilon_0 = \infty$ so that $P_{\epsilon_0}^a(t) = 1$). As in §2, we may define a multiplicative chaos which will be denoted by P^a, i.e., the weak* limit of $P_{\epsilon_n}^a(t) 1_{[0,1]}(t) dt$ (see [K4] for more details).

Many properties of the Poisson process are revealed by the kernel

$$k(t) = k_\mu(t) = \exp \varphi(t)$$

where

$$\varphi(t) = \int_{\mathbb{R}^+} (y - |t|)_+ d\mu(y)$$

($x_+ = \max(0, x)$ denoting the positive part of x).

Although many calculations in the sequel are valuable for arbitrary measure μ, we will concentrate on the measure

$$\mu = \sum_{n=1}^{\infty} \delta_{\ell_n} \quad \text{with } \ell_n = \alpha/n.$$

For this special case of μ, we have

$$k(t) = \exp \sum_{n=1}^{\infty} (\ell_n - |t|)_+ \approx 1/|t|^\alpha.$$

THEOREM 3: *Suppose* $\max(\alpha-1,0)<\beta<\alpha$. *If* $a=\beta/\alpha$, *then almost surely*

$$\dim P^a \geq 1-[\beta\log(\beta/\alpha)-(\beta-\alpha)].$$

The proof of this theorem is based on the following two propositions whose proofs will be postponed until §§6–7. The first concerns the variation of $N_{\ell_n}(t)$.

PROPOSITION 1: *We have almost surely*

$$\sup_{t,s\in(0,1);|t-s|\leq\ell_n}|N_{\ell_n}(t)-N_{\ell_n}(s)|=O\Big(\frac{\log n}{\log\log n}\Big).$$

The second concerns the tail measure R_n^a of P^a, where R_n^a is by definition the Poisson multiplicative chaos associated to the measure $\nu^{(\ell_n)}=\lambda\otimes\mu|_{(0,\ell_n)}$.

PROPOSITION 2: *Suppose* $0<a<1$ *and* $\alpha(1-a)<1$. *There is an almost surely finite variable* $0<C(\omega)<\infty$ *such that for any* $n\geq1$ *and for any interval* I *of length* ℓ_n *in* $[0,1]$, *we have*

$$R_n^a(I)\leq C(\omega)|I|\log\frac{1}{|I|}.$$

Proof of Theorem 3: Let $I_k(t)$ be the dyadic interval of the form $[i2^{-k},(i+1)2^{-k})$ containing t. Let n be the largest integer such that $\ell_n\geq 2^{-k}$. Then $\ell_{n+1}<2^{-k}$. Let $\tilde I_k(t)$ be an interval of length ℓ_n which contains $I_k(t)$. Since $\ell_n=\alpha/n$, we have

$$|\tilde I_k|\approx|I_k|\approx 2^{-k}\approx 1/n,\quad \nu_{\ell_n}(D_t)=\sum_{j=1}^n\ell_j\sim\alpha\log n.$$

Then by Proposition 1 and Proposition 2, a.s. for any $I_k(t)$ we have

$$\begin{aligned}P^a(I_k(t))\leq&P^a(\tilde I_k(t))\\=&\int_{\tilde I_k(t)}a^{N_{\ell_n}(s)}\exp((1-a)\nu_{\ell_n}(D_s))dR_n^a(s)\\=&O(n^{\alpha(1-a)}e^{O(\frac{\log n}{\log\log n})}a^{N_{\ell_n}(t)}R_n^a(\tilde I_k(t)))\\=&O\Big(n^{\alpha(1-a)}e^{O(\frac{\log n}{\log\log n})}a^{N_{\ell_n}(t)}|I_k(t)|\log\frac{1}{|I_k(t)|}\Big).\end{aligned}$$

Then

$$\liminf_{k\to\infty}\frac{\log P^a(I_k(t))}{\log|I_k(t)|}\geq 1-\alpha(1-a)-\log a\limsup_{n\to\infty}\frac{N_{\ell_n}(t)}{\log n}.$$

Let $Y_j=N(D_t\cap(\mathbb{R}\times\{\ell_j\}))$. Let $\mathcal{P}^a$ be the Peyrière measure, which is well defined because $\alpha(1-a)^2<\alpha(1-a)<1$. The variables $Y_j(\omega,t)$ are $\mathcal{P}^a$-independent.

We have $N_{\ell_n}(D_t) = \sum_{j=1}^n Y_j$ and

$$\mathbb{E}_{\mathcal{P}^a} Y_j = \mathbb{E}\frac{Y_j a^{Y_j}}{\mathbb{E}a^{Y_j}} = a\ell_j, \quad \mathbb{E}_{\mathcal{P}^a} Y_j^2 = \mathbb{E}\frac{Y_j^2 a^{Y_j}}{\mathbb{E}a^{Y_j}} = a\ell_j + (a\ell_j)^2.$$

It follows that a.s. P^a-a.e.

$$\lim_{n\to\infty} \frac{N_{\ell_n}(D_t)}{\log n} = a\alpha$$

(see the proof of Theorem 1 (a)). Thus a.s. P^a-a.e.

$$\liminf_{k\to\infty} \frac{\log P^a(I_k(t))}{\log |I_k(t)|} \geq 1 - \alpha(1-a) - a\alpha \log a.$$

By the formula at the end of §2, we have a.s.

$$\dim P^a \geq 1 - \alpha(1-a) - a\alpha\log a = 1 - (\alpha - \beta) - \beta \log(\beta/\alpha). \quad ∎$$

Following the idea in [K3], instead of $\{\ell_n\}$, we consider another sequence $\{\ell'_n\}$ which is constructed from $\{\ell_n\}$ as follows:

$$\ell'_{\frac{m(m-1)}{2}+1} = \cdots = \ell'_{\frac{m(m+1)}{2}} = \lambda_m \quad \text{with } \lambda_m = \ell_{\frac{m(m+1)}{2}}.$$

Evidently $\ell'_n \leq \ell_n$. More important is

$$\sum_{n=1}^{\infty} (\ell_n - \ell'_n) < \infty.$$

It follows that if $k'(\cdot)$ denotes the kernel associated to $\{\ell'_n\}$, $k'(t) \approx k(t)$. It will be easy to check that Propositions 1 and 2 and then Theorem 3 remain true if ℓ_n is replaced by ℓ'_n. Let Q'^a be the multiplicative chaos associated to ℓ'_n. We have the following equivalence result whose proof will be postponed until §8.

PROPOSITION 3: *Suppose $\int k(t)^{(1-a)^2} dt < \infty$ (that is $\alpha(1-a)^2 < 1$ when $\ell_n = \alpha/n$). Then almost surely, Q^a and Q'^a are equivalent.*

Now we compare Q'^a with a Poisson multiplicative chaos. A Poisson process with intensity ν may be constructed as follows. Fix the segment $J_{r,n} = [r, r+1] \times \{\ell_n\}$ ($r \in \mathbb{Z}, n \geq 1$). Let $N_{r,n}$ be a Poisson variable with mean value 1. The Poisson process with intensity $\nu|_{J_{r,n}}$ is the set of points $\{r + \eta_{r,n}^{(j)}\}_{1\leq j\leq N_{r,n}}$ where $\{\eta_{r,n}^{(j)}\}_{j\geq 1}$ is an i.i.d. sequence with uniform distribution on $[0,1]$, which is independent of $N_{r,n}$. The union of all such random sets, assumed independent, is a Poisson process with intensity ν.

We assume that $\ell_n = \ell'_n$. Recall that there is an i.i.d. sequence $\{\omega_j\}$ in the Dvoretzky covering model. We use these variables to construct the Poisson process. Let

$$N_m = \sum_{n=m(m-1)/2+1}^{m(m+1)/2} N_{0,n}$$

which is a Poisson variable with mean value m. Construct now the Poisson process on $[0,1]\times\{\lambda_m\}$: if $N_m \leq m$, we take the first N_m variables in $\{\omega_{m(m-1)/2+j} : 1 \leq j \leq m\}$ to be the variables $\eta_{0,n}^{(j)}$; if $N_m > m$, we take all variables in $\{\omega_{m(m-1)/2+j} : 1 \leq j \leq m\}$ and introduce $N_m - m$ supplementary variables. In the following theorem, by the Poisson process we mean this special Poisson process, which is closely related to the Dvoretzky covering.

Furthermore, we assume that $\ell_n \leq \delta$ for some small $0 < \delta < 1/2$. The proof of the following theorem will also be postponed until §8.

THEOREM 4: *Suppose $\ell'_n \leq \delta$ for some $\delta < 1/2$ and $\int k(t)^{(1-a)^2} dt < \infty$. Then $P'^a|_{[\delta,1]}$ and $Q'^a|_{[\delta,1]}$ are almost surely equivalent.*

Proof of Theorem 1 (c): In the proof of Theorem 1 (a), we see that F_β is a Borel support of Q^a for $a = \beta/\alpha$. So, $\dim Q^a \leq \dim F_\beta$. Then by Theorem 1 (b), we have

$$\dim Q^a \leq \dim F_\beta \leq 1 - [\beta \log(\beta/\alpha) - (\beta - \alpha)].$$

Therefore, we have only to show that

$$\dim Q^a \geq 1 - [\beta \log(\beta/\alpha) - (\beta - \alpha)]. \tag{5.1}$$

We are now going to prove (5.1) by showing that Q^a is equivalent to P'^a and that

$$\dim P'^a \geq 1 - [\beta \log(\beta/\alpha) - (\beta - \alpha)]. \tag{5.2}$$

Thus we will finish our proof.

We first remark that the Poisson multiplicative chaos P'^a is equivalent to its tails R'^a_n (it is the same for P^a and its tails R^a_n). In fact, if P'^a is defined by weights P'_j (we adopt the same notation as in [F1], a theorem in which it will be applied), then R^a_n may be defined by the weights P''_j with

$$P''_j = 1 \quad (1 \leq j \leq n), \quad P''_j = P'_j \quad (j > n).$$

Thus a simple application of the criterion in [F1] (Theorem 2.1) allows us to get the equivalence. The same argument shows that the multiplicative chaos Q'^a is also equivalent to its tails. Actually there is equivalence between the restrictions on any smaller interval of P'^a and its tails (and of Q^a and its tails).

By this remark, in order to compare P'^a and Q'^a (are they equivalent or not?), we have only to compare their tails, one tail of each. However, by Theorem 4, for any $0 < \delta < 1/2$, some tails of P'^a and Q'^a restriction on $[\delta, 1]$ are equivalent. Therefore $P'^a|_{[\delta,1]}$ is equivalent to $Q'^a|_{[\delta,1]}$ $(\forall\delta)$. Since almost surely, $P'^a(\{0\}) = 0$ and $Q'^a(\{0\}) = 0$, it follows that P'^a and Q'^a are equivalent on $[0, 1]$. Recall that Q'^a and Q^a are equivalent (Proposition 3). Thus Q^a and P'^a are equivalent.

The inequality (5.2) is true if $\dim P'^a$ is replaced by $\dim P^a$ (Theorem 3). Actually the very inequality (5.2) may be proved in the same way as Theorem 3. It suffices to note that

$$\ell'_n \sim \frac{\alpha}{n}, \quad \sum_{j=1}^{n} \ell'_j \sim \alpha \log n.$$

So, Proposition 1 and Proposition 2 remain true if ℓ_n is replaced by ℓ'_n. ∎

6. Variation of $N_{\ell_n}(t)$, Proof of Proposition 1

The following lemma will be useful.

LEMMA 1: *Let X be a Poisson variable with mean value m. For numbers m^- and m^+ such that $1 \le m^- < m < m^+ < \infty$, we have*

$$P(X \ge m^+) \le \exp\Big(-m + m^+ - m^+ \log \frac{m^+}{m}\Big),$$
$$P(X \le m^-) \le \exp\Big(-m + m^- - m^- \log \frac{m^-}{m}\Big).$$

Proof: By the Markov inequality,

$$P(X \ge m^+) \le e^{-\lambda m^+}\mathbb{E}(e^{\lambda X}) = \exp(-m^+\lambda + (e^\lambda - 1)m)$$

holds for all $\lambda > 0$. Now minimize over λ (the minimizing point is $\log(m^+/m)$). In the same way, we can prove the second inequality by using the fact that for all $\lambda > 0$,

$$P(X \le m^-) \le e^{\lambda}\mathbb{E}(e^{-\lambda X}). \quad ∎$$

The following corollary is an immediate consequence.

LEMMA 2: *Let $\frac{1}{2} < \delta < 1$ and $0 < \eta < 1$. There is a positive constant $M > 0$ such that for any Poisson variable X with mean value $m > M$ we have*

$$P(X < m - m^{\delta}) \leq \exp(-(1-\eta)m^{2\delta-1}),$$
$$P(X > m + m^{\delta}) \leq \exp(-(1-\eta)m^{2\delta-1}).$$

Proof of Proposition 1: It is clear that for any $\epsilon > 0$ we have

$$|N_\epsilon(t) - N_\epsilon(s)| \leq N_\epsilon(D_t \Delta D_s).$$

We cover the interval $[0,1]$ by $[\ell_n^{-1}] + 1$ intervals of length ℓ_n, which we denote by $\{J_k\}$ $(1 \leq k \leq [\ell_n^{-1}] + 1)$. Then

$$\sup_{t,s\in(0,1);|t-s|\leq \ell_n} N_{\ell_n}(D_t \Delta D_s) \leq \sup_k \sup_{t\in J_k;|t-s|\leq\ell_n} N_{\ell_n}(D_t\Delta D_s).$$

If $|s-t| \leq \ell_n$, we have

$$\sup_{t\in J_k;|t-s|\leq \ell_n} N_{\ell_n}(D_t\Delta D_s) \leq N_{\ell_n}(D_k)$$

where

$$D_k = \bigcup_{t\in J_k, |t-s|\leq \ell_n} D_t \Delta D_s.$$

Notice that D_k is a union of two strips (not disjoint); one is vertical and the other is oblique with slope -1 but both are of width $3\ell_n$. Hence

$$\nu_{\ell_n}(D_k) = \sum_{j=1}^{n} \lambda \otimes \delta_{\ell_j}(D_k) \leq 2\cdot(3\ell_n)\sum_{j=1}^{n} 1 = 6\ell_n n = O(1).$$

Apply Lemma 1 to $m^+ = \tau\frac{\log n}{\log\log n}$ with $\tau > 0$ sufficiently large. For $\eta > 0$, we have

$$\begin{aligned} P\Big(\sup_{t,s\in(0,1);|t-s|\leq\ell_n} N_{\ell_n}(D_t\Delta D_s) \geq m^+\Big) &\leq \sum_k P(N_{\ell_n}(D_k) \geq m^+) \\ &\leq O(\ell_n^{-1}\exp(-(1-\eta)m^+\log m^+)) \\ &\leq O\Big(\frac{1}{n^{(1-\eta)^2\tau-1}}\Big). \end{aligned}$$

Take τ and η such that $(1-\eta)^2\tau > 2$. Then apply the Borel-Cantelli lemma. ∎

7. Tail measure R_n^a, Proof of Proposition 2

The kernel associated to the truncated measure μ_ϵ is denoted by k_ϵ. The corresponding function φ will be denoted by φ_ϵ

For $p \geq 2$ and p points $t_1, \ldots, t_p$ on $\mathbb{R}$, denote

$$\Phi_\epsilon(t_1, \ldots, t_p) = \mathbb{E} \prod_{j=1}^{p} P_\epsilon^a(t_j).$$

For any finite interval I, we have

$$\mathbb{E}(P_\epsilon^a(I))^p = \int_I \cdots \int_I \Phi_\epsilon(t_1, \ldots, t_p) dt_1 \cdots t_p.$$

So, in order to estimate $P_\epsilon^a(I)$, we are led to estimate Φ_ϵ.

LEMMA 3: *Suppose $t_1 \geq t_2 \geq \cdots \geq t_{p-1} \geq t_p$. Then*

$$\Phi_\epsilon(t_1, \ldots, t_p) = \prod_{i=0}^{p-2} \left(\prod_{j=1}^{p-i-1} k_\epsilon(t_{j+i+1} - t_j) \right)^{a^i(1-a)^2}.$$

Proof: For $1 \leq j \leq p$ and $1 \leq k \leq j$, let

$$D_{j,k} = \{(x,y) : t_{j+1} \leq x \leq t_j, -x + t_{j-(k-1)} \leq y \leq -x + t_{j-k}\}$$

(with convention $t_{p+1} = -\infty$ and $t_0 = +\infty$). This is a partition of $\bigcup_{j=1}^p D_{t_j}$ and points in $D_{j,k}$ are repeated k times. Then we have

$$\sum_{j=1}^{p} N_\epsilon(D_{t_j}) = \sum_{j=1}^{p} \sum_{k=1}^{j} k N_\epsilon(D_{j,k}).$$

Since $N_\epsilon(D_{j,k})$'s are independent Poisson variables, we have

$$\mathbb{E} a^{\sum_{j=1}^p N_\epsilon(D_{t_j})} = \exp\left[\sum_{j=1}^{p} \sum_{k=1}^{j} \nu_\epsilon(D_{j,k})(a^k - 1) \right].$$

Also we have

$$\prod_{j=1}^{p} \exp[(a-1)\nu_\epsilon(D_{t_j})] = \exp\left[(a-1) \sum_{j=1}^{p} \sum_{k=1}^{j} k \nu_\epsilon(D_{j,k}) \right].$$

By combining these, we get the expression

$$\Phi_\epsilon(t_1, \ldots, t_p) = \exp\left[\sum_{j=1}^{p} \sum_{k=1}^{j} a_k \nu_\epsilon(D_{j,k}) \right]$$

where $a_k = (a^k - 1) - k(a-1)$. Notice that $a_1 = 0$.

Now remark that

$$\begin{aligned}\nu_\epsilon(D_{j,k}) =&\nu_\epsilon(D_{t_j} \cap D_{t_{j-(k-1)}}) - \nu_\epsilon(D_{t_{j+1}} \cap D_{t_{j-(k-1)}}) \\ &- \nu_\epsilon(D_{t_j} \cap D_{t_{j-k}}) + \nu_\epsilon(D_{t_{j+1}} \cap D_{t_{j-k}}).\end{aligned}$$

A simple calculation gives $\nu_\epsilon(D_t \bigcap D_s) = \varphi(t-s)$. So we have

$$\nu_\epsilon(D_{j,k}) = \varphi_\epsilon(t_j - t_{j-(k-1)}) - \varphi_\epsilon(t_{j+1} - t_{j-(k-1)}) - \varphi_\epsilon(t_j - t_{j-k}) + \varphi_\epsilon(t_{j+1} - t_{j-k}).$$

(with the convention that $\varphi(\infty) = 0$, so that for $\nu_\epsilon(D_{j,j})$ the third and fourth terms are zero; for $\nu_\epsilon(D_{p,p})$, even the second term is zero; for $\nu_\epsilon(D_{p,k})$ with $1 \le k < p$, the second and fourth terms are zero).

We are ready to prove the expression of $\Phi(t_1, \ldots, t_p)$ by induction on p. The case $p = 2$ was known [FK1]. Suppose the expression is true for p. Notice that the partition $\{D_{j,k}\}$ depends on p. To distinguish, we use $\{D'_{j,k}\}$ to denote the corresponding partition for $p+1$. We remark that $D'_{j,k} = D_{j,k}$ for $1 \le j < p$. So, if we let

$$F_\epsilon(t_1, \ldots, t_p) = \log \Phi_\epsilon(t_1, \ldots, t_p),$$

we have

$$F_\epsilon(t_1, \ldots, t_p, t_{p+1}) - F_\epsilon(t_1, \ldots, t_p) = \sum_{j=p}^{p+1} \sum_{k=1}^{j} a_k \nu_\epsilon(D'_{j,k}) - \sum_{k=1}^{p} a_k \nu_\epsilon(D_{p,k}).$$

When an extra point t_{p+1} is added, $D_{p,k}$ changes as follows:

$$D_{p,k} = D'_{p,k} \cup D'_{p+1,k+1} \quad (1 \le k \le p).$$

There is again a new portion $D'_{p+1,1}$, but it doesn't contribute to $F_\epsilon(t_1, \ldots, t_{p+1})$ because of $a_1 = 0$. Note that we have

$$a_1 \nu_\epsilon(D'_{p,1}) + a_2 \nu_\epsilon(D'_{p+1,2}) - a_1 \nu_\epsilon(D_{p,1}) = a_2 [F_\epsilon(t_{p+1} - t_p) - F_\epsilon(t_{p+1} - t_{p-1})]$$

and, for $2 \le k \le p$, we have

$$\begin{aligned}&a_k \nu_\epsilon(D'_{p,k}) + a_{k+1} \nu_\epsilon(D'_{p+1,k+1}) - a_k \nu_\epsilon(D_{p,k}) \\ &\quad = (a_{k+1} - a_k)[F_\epsilon(t_{p+1} - t_{p-(k-1)}) - F_\epsilon(t_{p+1} - t_{p-k})].\end{aligned}$$

Adding these p equalities gives

$$\begin{aligned}&F_\epsilon(t_1, \ldots, t_p, t_{p+1}) - F_\epsilon(t_1, \ldots, t_p) \\ &\quad = a_2 F_\epsilon(t_{p+1} - t_p) + \sum_{k=3}^{p+1} (a_k - 2a_{k-1} + a_{k-2}) F_\epsilon(t_{p+1} - t_{p-(k-2)}).\end{aligned}$$

Notice that

$$a_2 = (1-a)^2, \quad a_k - 2a_{k-1} + a_{k-2} = a^{k-2}(1-a)^2 \quad (3 \le k \le p+1).$$

It follows that

$$\frac{\Phi_\epsilon(t_1, \ldots, t_{p+1})}{\Phi_\epsilon(t_1, \ldots, t_p)} = \prod_{k=1}^{p} k_\epsilon(t_{p+1} - t_{p-(k-1)})^{a^{k-1}(1-a)^2}.$$

Thus the induction is finished. ∎

Suppose $0 < a < 1$. Let

$$q_r = \frac{1-a^{p-1}}{1-a} \cdot \frac{1}{a^{r-1}} \quad (1 \le r \le p-1).$$

We have $\sum_{r=1}^{p-1} q_r^{-1} = 1$. Let $I = [c, d]$. By the Hölder inequality, we get that $\mathbb{E}(P_\epsilon^a(I))^p$ is bounded by

$$\begin{aligned} p! & \int \cdots \int_{c \le t_p \le \cdots \le t_2 \le t_1 \le d} \Phi_\epsilon(t_1, \ldots, t_p) dt_1 \cdots dt_p \\ \le p! & \prod_{r=1}^{p-1} \left(\int \cdots \int_{c \le t_p \le \cdots \le t_2 \le t_1 \le d} \prod_{j=1}^{p-r} k_\epsilon(t_{j+r} - t_j)^{(1-a)(1-a^{p-1})} dt_1 \cdots dt_p \right)^{1/q_r} \\ \le p! & \int \cdots \int_{c \le t_p \le \cdots \le t_2 \le t_1 \le d} \prod_{j=1}^{p-1} k_\epsilon(t_{j+1} - t_j)^{1-a} dt_1 \cdots dt_p. \end{aligned}$$

We have used the facts that $k_\epsilon(t) \ge 1$ and $k_\epsilon(t_{j+r} - t_j) \le k_\epsilon(t_{j+1} - t_j)$ because $k_\epsilon(t)$ is symmetric and decreasing on $(0, \infty)$. Thus we obtain

LEMMA 4: *Suppose $0 < a < 1$. For any interval $I = [c, d]$ and any integer $p \ge 1$, we have*

$$\mathbb{E}(P_\epsilon^a(I))^p \le p! \int \cdots \int_{c \le t_p \le \cdots \le t_2 \le t_1 \le d} \prod_{j=1}^{p-1} k_\epsilon(t_{j+1} - t_j)^{1-a} dt_1 \cdots dt_p.$$

LEMMA 5: *Suppose $\ell_n = \gamma/n$ with $0 < \gamma < 1$. For any $\beta \ge 0$ and any $0 < s \le \ell_n$, we have*

$$\int_0^s (s-u)^\beta \exp \sum_{j=n}^{\infty} (\ell_j - u)_+ du \le C \mathrm{B}(1-\gamma, 1+\beta) \ell_n^\gamma s^{1-\gamma+\beta}$$

where $\mathrm{B}(\cdot,\cdot)$ *is the Beta function and* C *is a constant independent of* γ, β *and* n.

Proof: Introduce the change of variable $u = vs$. The integral to be bounded becomes

$$s^{1+\beta}\int_0^1 (1-v)^\beta \exp\sum_{j=n}^{\infty}(\ell_j - sv)_+ dv.$$

For fixed v, define $m(v)$ as the integer for which $m(v) < \gamma/sv \leq m(v)+1$. Then the sum $\sum_{j=n}^{\infty}(\ell_j - sv)_+$ is bounded by

$$\sum_{j=n}^{m(v)} \ell_j < \gamma \log \frac{m(v)}{n-1} < \gamma \log \frac{\gamma}{sv(n-1)} < \gamma \log \frac{\ell_n}{sv} + \log 2$$

for $n \geq 2$, whence the integral in question is bounded by

$$2s^{1+\beta-\gamma}\ell_n^\gamma \int_0^1 (1-v)^\beta v^{-\gamma} dv.$$

That proves the inequality for $n \geq 2$. Notice that $\exp\sum_{j=1}^{\infty}(\ell_j - u)_+ \approx u^{-\gamma}$. A direct calculation shows that the inequality also holds for $n = 1$ if $C \geq 2$ is chosen sufficiently large. ∎

Proof of Proposition 2: Recall that the kernel associated to $\{\ell_j\}_{j>n}$ is

$$k^{(n)}(t) = \exp\sum_{j=n+1}^{\infty}(\ell_j - |t|)_+.$$

Then we have $k^{(n)}(t)^{1-a} = \tilde{k}^{(n)}((1-a)t)$ where $\tilde{k}^{(n)}$ is the kernel associated to $\{(1-a)\ell_j\}_{j>n}$. Let $I = [c,d]$ with $|I| = \ell_{n+1}$. By Lemma 4 and the Fatou lemma, we get that $\mathbb{E}(R_n^a(I))^p$ is bounded by

$$\begin{aligned} &p! \int\cdots\int_{c\leq t_1\leq\cdots\leq t_p\leq d} \prod_{j=1}^{p-1} \tilde{k}^{(n)}((1-a)(t_{j+1}-t_j))dt_1\cdots dt_p \\ =&\frac{p!}{(1-a)^p} \int\cdots\int_{c(1-a)\leq t_1\leq\cdots\leq t_p\leq(1-a)d} \prod_{j=1}^{p-1}\tilde{k}^{(n)}(t_{j+1}-t_j)dt_1\cdots dt_p. \end{aligned}$$

The last integral is equal to

$$\int\cdots\int_{0\leq s_1\leq s_2\leq\cdots\leq s_p\leq(1-a)\ell_{n+1}} \prod_{j=1}^{p-1}\tilde{k}^{(n)}(s_{j+1}-s_j)ds_1\cdots ds_p$$

$$= \int\cdots\int_{u_1\geq 0,\cdots,u_p\geq 0; u_1+\cdots+u_p\leq \tilde{\ell}_{n+1}} \tilde{k}^{(n)}(u_2)\cdots\tilde{k}^{(n)}(u_p)du_1\cdots du_p$$

$$= \int_0^{\tilde{\ell}_{n+1}}\int_0^{\tilde{\ell}_{n+1}-u_1}\tilde{k}^{(n)}(u_2)\int_0^{\tilde{\ell}_{n+1}-(u_1+u_2)}\cdots$$

$$\cdots\int_0^{\tilde{\ell}_{n+1}-(u_1+\cdots+u_{p-2})}\tilde{k}^{(n)}(u_{p-1})$$

$$\times\int_0^{\tilde{\ell}_{n+1}-(u_1+\cdots+u_{p-1})}\tilde{k}^{(n)}(u_p)du_p du_{p-1}\cdots du_2 du_1$$

where $\tilde{\ell}_{n+1} = (1-a)\ell_{n+1} = (1-a)|I|$. We have made changes of variables. Denote

$$\gamma = (1-a)\alpha.$$

By Lemma 5, the last integral is bounded by

$$C(1-a)^\gamma \mathrm{B}(1-\gamma,1)|I|^\gamma(\tilde{\ell}_{n+1}-(u_1+\cdots+u_{p-1}))^{1-\gamma}$$

where C is the constant in Lemma 5. Substitute this into the integral of the next to last integral. Then, using once more the preceding lemma, the last double integral is bounded by

$$C^2(1-a)^{2\gamma}\mathrm{B}(1-\gamma,1)\mathrm{B}(1-\gamma,1+(1-\gamma))|I|^{2\gamma}(\tilde{\ell}_{n+1}-(u_1+\cdots+u_{p-1}))^{2(1-\gamma)}.$$

Inductively we get that the initial integral is bounded by

$$C^{p-1}(1-a)^p|I|^p\prod_{j=1}^{p-1}\mathrm{B}(1-\gamma,1+(j-1)(1-\gamma)).$$

By the formula $\mathrm{B}(p,q)=\Gamma(p)\Gamma(q)/\Gamma(p+q)$ where Γ is the Gamma function, we have

$$\prod_{j=1}^{p-1}\mathrm{B}(1-\gamma,1+(j-1)(1-\gamma)) = \frac{\Gamma(1-\gamma)^{p-1}}{\Gamma(1+(p-1)(1-\gamma))}.$$

Therefore, there is some constant C (not the same as that of Lemma 5) such that

$$\mathbb{E}(R_n^a(I))^p \leq \frac{p!(C|I|)^p}{\Gamma(1+(1-\gamma)(p-1))}.$$

It can be deduced that there are constants $A>0$ and $B>0$ such that for any $\xi>0$ we have

$$\mathbb{E}e^{\xi R_n^a(I)} \leq A\exp(B(\xi|I|)^{1/(1-\gamma)}).$$

It follows that the inequality

$$\mathbb{E}e^{\xi R_n^a(I)} \leq A\exp(B(\xi\ell_{n+1})^{1/(1-\gamma)})$$

holds for any interval I such that $|I| \leq \ell_{n+1}$ (we deduce it from the preceding one using the monotonicity of the measure R_n^a). Now divide $[0,1]$ into intervals $\mathcal{J}_n := \{J_1, \ldots, J_{k(n)}\}$ where

$$J_m = [\alpha(m-1)/(n+1), \alpha m/(n+1)] \quad (1 \leq m < k(n)),$$
$$J_{k(n)} = [\alpha(k(n)-1)/(n+1), 1].$$

Notice that $\alpha(k(n)-1) < n+1 \leq \alpha k(n)$. Then for arbitrary $r > 0$, for any $J \in \mathcal{J}_n$, we apply the above inequality with $\xi = \ell_{n+1}^{-1}$ to get

$$\begin{aligned} &P(R_n^a(J) \geq r\ell_{n+1}\log\ell_{n+1}^{-1}) \\ \leq &A\exp(B(\xi\ell_{n+1})^{1/(1-\gamma)} - r\xi\ell_{n+1}\log\ell_{n+1}^{-1}) = O\Big(\frac{1}{n^r}\Big). \end{aligned}$$

Hence

$$\begin{aligned} P(\max_{J\in\mathcal{J}_n} R_n^a(J) \geq r\ell_{n+1}\log\ell_{n+1}^{-1}) \leq &\sum_{J\in\mathcal{J}_n} P(R_n^a(J) \geq r\ell_{n+1}\log\ell_{n+1}^{-1}) \\ =&O\Big(\frac{k(n)}{n^r}\Big) = O\Big(\frac{1}{n^{r-1}}\Big) \end{aligned}$$

since $k(n) \leq 1+(n+1)/\alpha$. Take $r > 2$. We have, from the Borel–Cantelli lemma, that

$$C(\omega) := \sup_{n\in\mathbb{N}} \max_{J\in\mathcal{J}_n} \frac{R_n^a(J)}{\ell_{n+1}\log\ell_{n+1}^{-1}} < \infty$$

with probability equal to one. Now it suffices to observe that any interval of length ℓ_n can be covered by at most three intervals in $\mathcal{J}_n$. ∎

8. Equivalence of Q^a and P^a

Proof of Proposition 3: By a criterion in [F1] (Theorem 2.1 (a)), it suffices to show that

$$\prod_{n=1}^{\infty}\mathbb{E}\sqrt{\frac{a^{\chi_n(t-\omega_n)+\chi'_n(t-\omega_n)}}{(1+(a-1)\ell_n)(1+(a-1)\ell'_n)}} > 0$$

where χ' is the characteristic function of the interval $(0,\ell'_n)$. The distribution of $X = \chi_n(t-\omega_n) + \chi'_n(t-\omega_n)$ does not depend on t. We have actually

$$P(X=0) = 1-\ell_n, \quad P(X=1) = \ell_n - \ell'_n, \quad P(X=2) = \ell'_n.$$

Thus, we have

$$\mathbb{E}\sqrt{a^X} = \exp[-\ell_n + \sqrt{a}(\ell_n - \ell'_n) + a\ell'_n + O(\ell_n^2)].$$

On the other hand,

$$\sqrt{(1+(a-1)\ell_n)(1+(a-1)\ell'_n)} = \exp\Big[\frac{a-1}{2}(\ell_n + \ell'_n) + O(\ell_n^2)\Big].$$

Then the general term in the above infinite product equals

$$\exp\Big[-\frac{1}{2}(\sqrt{a}-1)^2(\ell_n - \ell'_n) + O(\ell_n^2)\Big].$$

Since $\sum(\ell_n - \ell'_n) < \infty$ and $\sum \ell_n^2 < \infty$, the infinite product is positive. ∎

Proof of Theorem 4: Since $\ell'_n \le \delta$, the multiplicative chaos P'^a restricted on $[\delta, 1]$ involves only the random points with abscissa in $[0,1]$. Let $1_{(0,\lambda_m)}(t)$ be the characteristic function of the interval $(0,\lambda_m)$, which is periodically extended on $\mathbb{R}$ with period 1 so that it can be considered as a function on $\mathbb{T}$. Let

$$S_m(t) = \sum_{j=1}^{m} 1_{(0,\lambda_m)}(t - \omega_{m(m-1)/2+j}),$$

and

$$\mathcal{N}_m(t) = N(D_t \cap (\mathbb{R}\times\{\lambda_m\})) = \sum_{j=1}^{N_m} 1_{(0,\lambda_m)}(t - \eta_{m,j})$$

where N_m is a Poisson variable with mean value m, $\eta_{m,j} = \omega_{m(m-1)/2+j}$ for $1 \le j \le m$ and $\eta_{m,j}$ for $j > m$ are also uniformly distributed. All variables N_m and $\eta_{m,j}$ are independent. By the same criterion we used for proving Proposition 3, it suffices to show that

$$\prod_{m=1}^{\infty} \mathbb{E}\sqrt{\frac{a^{S_m(t)+\mathcal{N}_m(t)}}{(1+(a-1)\lambda_m)^m e^{(a-1)m\lambda_m}}} > 0.$$

First notice that

$$(1+(a-1)\lambda_m)^m \le \exp((a-1)m\lambda_m).$$

Then we are led to estimate $E := \mathbb{E}\sqrt{a^{S_m(t)+\mathcal{N}_m(t)}}$.

$$\begin{aligned}
E =& e^{-m}\sum_{n=0}^{\infty}\frac{m^n}{n!}\mathbb{E}a^{\frac{1}{2}\sum_{j=1}^{m}1_{(0,\lambda_m)}(t-\omega_{m(m-1)/2+j})+\frac{1}{2}\sum_{j=1}^{n}1_{(0,\lambda_m)}(t-\eta_{m,j})}\\
=& e^{-m}\sum_{n=0}^{m}\frac{m^n}{n!}\mathbb{E}a^{\sum_{j=1}^{n}1_{(0,\lambda_m)}(t-\eta_{m,j})+\frac{1}{2}\sum_{j=n+1}^{m}1_{(0,\lambda_m)}(t-\omega_{m(m-1)/2+j})}\\
&+e^{-m}\sum_{n=m+1}^{\infty}\frac{m^n}{n!}\mathbb{E}a^{S_m(t)+\frac{1}{2}\sum_{j=m+1}^{n}1_{(0,\lambda_m)}(t-\eta_{m,j})}\\
=:& e^{-m}\left[\sum_{n=0}^{m}\frac{m^n}{n!}A_{m,n}+\sum_{n=m+1}^{\infty}\frac{m^n}{n!}B_{m,n}\right]
\end{aligned}$$

where

$$A_{m,n}=(1+(a-1)\lambda_m)^n(1+(\sqrt{a}-1)\lambda_m)^{m-n},$$
$$B_{m,n}=(1+(a-1)\lambda_m)^m(1+(\sqrt{a}-1)\lambda_m)^{n-m}.$$

Take $1/2<\epsilon<1$. There is a constant $c_1>0$ such that for $m-m^\epsilon\le n\le m$,

$$(1+(\sqrt{a}-1)\lambda_m)^{m-n}\ge e^{-c_1\lambda_m m^\epsilon},$$

and for $m<n\le m+m^\epsilon$,

$$\left(\frac{1+(\sqrt{a}-1)\lambda_m}{1+(a-1)\lambda_m}\right)^{n-m}\ge e^{-c_1\lambda_m m^\epsilon}.$$

So both $A_{m,n}$ (when $m-m^\epsilon\le n\le m$) and $B_{m,n}$ (when $m<n\le m+m^\epsilon$) are bounded from below by

$$(1+(a-1)\lambda_m)^n\cdot e^{-c_1\lambda_m m^\epsilon}.$$

It follows that

$$\begin{aligned}
E \ge& e^{-c_1\lambda_m m^\epsilon}\cdot e^{-m}\sum_{n=m-m^\epsilon}^{m+m^\epsilon}\frac{(m(1+(a-1)\lambda_m))^n}{n!}\\
=& e^{-c_1\lambda_m m^\epsilon}e^{(a-1)m\lambda_m}[1-P(Y<m-m^\epsilon\text{ or }Y>m+m^\epsilon)]
\end{aligned}$$

where Y denotes a Poisson variable with mean value $m(1+(a-1)\lambda_m)\sim m$. By Lemma 2, the probability $P(\cdot)$ in the last expression is bounded by $e^{-c_2m^{2\epsilon-1}}$ for some $c_2>0$. Thus, the infinite product is bounded from below by

$$\exp\left[-c_1\sum_{m=1}^{\infty}\lambda_m m^\epsilon\right]\prod_{m=1}^{\infty}(1-e^{-c_2m^{2\epsilon-1}})>0. \qquad ∎$$

9. Final remarks

1. The method that we have used to prove Proposition 2, i.e., estimating the Laplace transform of $R_n^a(I)$ and then using it to bound probabilities, does not work when $a > 1$, as the Laplace transform of $R_n^a(I)$ is infinite for all $\xi > 0$ in this case. Actually we have $\mathbb{E}(R_n^a(I))^p = \infty$ for large p. In fact, by Lemma 3, for any interval $I = [c, d]$ we have

$$\mathbb{E}(P_\epsilon^a(I))^p \geq p! \int\cdots\int_{c\leq t_p\leq\cdots\leq t_1\leq d} k_\epsilon(t_p - t_1)^\tau dt_1 \cdots t_p$$

where $\tau = a^{p-2}(1-a)^2$. Introducing changes of variables, we have

$$\begin{aligned}\mathbb{E}(P_\epsilon^a(I))^p \geq & p! \int\cdots\int_{u_1+\cdots+u_p\leq|I|} k_\epsilon(u_2+\cdots+u_p)^\tau du_1\cdots u_p \\ \geq & p!\int_0^{|I|} k_\epsilon(v_p)^\tau dv_p \int\cdots\int_{v_1+\cdots+v_{p-1}\leq|I|-v_p} dv_1\cdots v_{p-1} \\ = & \frac{p!}{(p-1)!}\int_0^{|I|} k_\epsilon(v_p)^\tau(|I|-v_p)^{p-1}dv_p.\end{aligned}$$

(For simplicity, we do not mark $u_j \geq 0$ and $v_j \geq 0$ in the domains of integration). It follows that if $\alpha a^{p-2}(1-a)^2 \geq 1$ (recall that $\ell_n = \alpha/n$), the last integral tends to infinity as $\epsilon \to 0$. Therefore the martingale $P_\epsilon^a(I)$ doesn't converge in L^p for large p. It is equivalent to saying that $\mathbb{E}(P^a(I))^p = \infty$ or $\mathbb{E}(R_n^a(I))^p = \infty$. However, as we have seen, when $0 < a < 1$, the same martingale converges in L^p for all p. That is an essential difference between the case $0 < a < 1$ and the case $a > 1$.

2. In [F3], there is a discussion of L^p convergence in the case $a = 1$, where the martingale is suitably defined.

3. If we had a better estimation than that in Lemma 4, the restriction $\max(\alpha - 1, 0) < \beta$ would be relaxed. But notice that there is nothing better to do when $\alpha < 1$ for $\max(\alpha - 1, 0) = 0$.

4. Instead of $\mathbb{E}\prod_{j=1}^p P_\epsilon^a(t_j)$, we consider $\mathbb{E}\prod_{j=1}^p Q_n^a(t_j)$. Finding a formula similar to that in Lemma 3 seems more difficult. This is why we call for Poisson multiplicative chaos. The cost is the equivalence theorem (Theorem 4).

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